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Dynamic System Characterization via Eigenvalue Orbits

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A new model-free approach for the description of general dynamical systems with unknown structure, order, and excitation is introduced. The approach is based on the new concept of eigenvalue orbit. The eigenorbits are obtained by building an associated linear time-variant system through a matrix that relates the output measurements in a moving horizon window and viewing the trajectories of its time-varying eigenvalues. How the eigenorbits may be computed from the measurements and used for the characterization of the original system is shown. The basic properties of the eigenorbits are presented via a series of theorems for the case of a discrete-time, linear time-invariant system. A set of examples are included to illustrate these properties for more general classes of systems and to suggest some practical issues that can be drawn from the orbits.

Introduction

CONSIDERABLE progress has been made in the fields of modeling and control of dynamic systems, and both disciplines have reached a high level of maturity and sophistication. The abundant literature on these subjects, however, is based on the unstated implication that the solution of a real control problem is the superposition of two separate tasks, namely, that of modeling and control. It has been shown by Skelton[1] that these two processes are, in reality, not separable. Models, no matter how sophisticated and elaborate, are always incomplete and never exactly describe the physical phenomena. Models can be built based on a set of assumptions, decided on by the analyst, that form the idealization of the system and known physical laws to formulate a mathematical model of the idealized system.

An alternative way to build a model is the use of identification methods relating input/output data.[2] An important contribution to the identification of modal properties of linear systems under free-response conditions has been given by Juang and Pappa[3] in their work on the eigensystem realization algorithm (ERA). The linearization properties of ERA have been employed in identifying approximate linear models for simple nonlinear systems by Horta and Juang.[4] An interesting method based on ERA for identifying nonlinear interactions in structures is presented by Balachandran et al.[5] According to the authors, using the ERA in conjunction with a sliding time-windowing technique reveals oscillating damping coefficients when nonlinear coupling (interaction) between structural modes is present. This method shows how linear identification algorithms may be used to capture nonlinear phenomena within a dynamical process. Examples of quadratically and cubically coupled pairs of oscillators illustrate the performance of the algorithm.

It is clear that the majority of the modern control-oriented work is actually based on some sort of model. Sometimes, the plant model is augmented with a disturbance model so as to take into account the information that is available about the environment while synthesizing the controller. On other occasions, the uncertainties present within the system are explicitly taken into account, and a wide variety of methods have been developed to cope with stochasticity, time dependence, parameter jumps, etc. However, no matter how sophisticated the model, or its treatment, the model is nearly always the central issue of practically any control problem.

In recent years, approaches not relying on concepts of the described classes have been proposed for the description and control of dynamic systems in a model-free perspective. Neural networks and fuzzy logic are nowadays well known within this context; see Refs. 6 and 7.

This paper proposes a novel and general model-free approach to the problem of general process characterization and control of dynamical systems. This description is based on the concept of eigenvalue orbits. These orbits are built by graphing the paths of the eigenvalues of a dynamic matrix, which relates the output measurements supplied by a set of sensors within a moving-time window. No assumptions on system linearity, order, parameter uncertainties, or operating environment shall be formulated or used toward the end of model synthesis. The concept of model is, in the framework of the present work, left as such. The only information that shall be used for the purpose of providing process description is supplied by the sampled sensor readings.

The approach presented herein for system characterization shares certain common points and analogies with the ERA.[5] In fact, it can be also considered as a kind of linearization method attempting to capture information from nonlinear systems by linear tools. However, in a different vein, this paper is not concerned with explicitly identifying structural parametrical properties of the system but just with drawing the trajectories of the eigenvalues of a time-variant matrix practically built from sensor readings. These trajectories, as is shown in the following sections, contain interesting information on the system being described.

The theoretical framework of the eigenvalue orbits is related to such concepts as phase-portraits, Lyapunov exponents, and Lyapunov transformations.[8] A more in-depth treatment of these issues is covered in Refs. 9 and 10. However, it appears that in the fields of dynamic systems, system identification, or model-free control practically no literature on eigenvalue orbits exists.

The main objective of this paper is to introduce the concept of eigenvalue orbit as a model-free descriptor of dynamic systems. Next, the basic eigenorbit theorems are demonstrated in the case of discrete-time, linear time-invariant (DLTI) systems. Finally, a set of numerical experiments are presented with the scope of highlighting the basic properties of the orbits and suggesting some practical features of the system behavior that can be drawn from the orbits.

Background Concepts

Consider a general nonlinear system of the type

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, t), \qquad \boldsymbol{y} = g(\boldsymbol{x}, t) \tag{1}$$

with $\boldsymbol{x} \in R^n$ and $\boldsymbol{y} \in R^q$, where n is the number of states and q the number of measurement channels, and where the structure of both f and g is supposed to be unknown, together with the order of $\boldsymbol{x}$. We base the central idea behind the schemes proposed on the measurements $\boldsymbol{y}$ being seen as exact and direct images of the states of some unknown dynamical system. The sensor outputs are a superposition

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of the physical variable we want to observe and measurement noise. However, to actually operate this distinction often implies great effort from a modeling and analytical point of view, and in some cases (see Ref. 3) the results may be questionable. Separation of plant and noise modes sometimes depends on the selection of some specific thresholds, such as the cutoff singular values, and is often related to purely numerical issues. The classical approach to a generic control problem is, in general terms, based on some sort of a model of the process one wishes to control, and, subsequently, on the selection of an appropriate controller. The approach advocated herein is to replace the plant model with a set of eigenorbits, which, as will be shown, capture all of the information on the process, including the measurement noise, and provide an alternative and generic description of any dynamic system.

Let us consider that the sensor outputs are sampled with a sampling time τ and that, at time t, one may assemble the following $q \times q$ matrices:

$$Y(t) = [\boldsymbol{y}(t), \boldsymbol{y}(t-\tau), \ldots, \boldsymbol{y}(t-(q-1)\tau)] \tag{2}$$

and

$$\dot{Y}(t) = [\dot{\boldsymbol{y}}(t), \dot{\boldsymbol{y}}(t-\tau), \ldots, \dot{\boldsymbol{y}}(t-(q-1)\tau)] \tag{3}$$

Let us relate the preceding two matrices in the form

$$\dot{Y}(t) = H(t)Y(t) \tag{4}$$

The $q \times q$ matrix $H(t)$ is, in practice, a fit to the sensor readings gathered in $Y(t)$ and $\dot{Y}(t)$, and the approach can be interpreted as a linearization of the original nonlinear state equations over a time horizon of length $q\tau$, with $H(t)$ playing the role of a Jacobian matrix. Because the state equation of the original system is unknown, our aim is to use the information available from the sensors to compute matrix H at each time t. According to the preceding equation, we may write

$$H(t) = \dot{Y}(t)Y(t)^{-1} \tag{5}$$

In case of ill conditioning of $Y(t)$, one may include more measurement instants than sensors. In this case, $Y(t)$ becomes rectangular and singular value decomposition may be used to obtain a pseudoinverse

$$H(t) = \dot{Y}(t)Y(t)^{+} \tag{6}$$

with $H(t) \in R^{q,q}$ in either case.

At each time t we may consider the following eigenvalue problem:

$$H(t)\Phi(t) = \Phi(t)\Lambda(t) \tag{7}$$

where $\Lambda(t) = \mathrm{diag}[\lambda_i(t)]$ is the eigenvalue matrix and $\Phi(t)$ the eigenmode matrix of $H(t)$. In general, the eigenvalues shall be complex, with

$$\lambda_i(t) = \Re[\lambda_i(t)] + j\Im[\lambda_i(t)] = u_i(t) + jv_i(t) \tag{8}$$

where $j = \sqrt{-1}$ and u_i and v_i are functions of time with $i = 1, 2, \ldots, q$. Interpreting now u_i and v_i as coordinates in the phase plane, one can view each eigenvalue of $H(t)$ as satisfying the dynamics of a (generally) nonlinear system of the form

$$v_i = \phi_i(u_i) \tag{9}$$

In other words, the real part of each eigenvalue of $H(t)$ may be seen as a displacement and the imaginary part as a velocity of a certain second-order system. The plot u–v is termed eigenvalue orbit. Displacement-velocity plots are known in the literature as phase portraits. When a phase portrait corresponds to a closed periodic curve, it is called orbit. In analogy with classical phase portraits, closed eigenvalue orbits may be referred to as eigenportraits. As detailed in subsequent sections, the eigenvalue orbits will play an essential role in characterizing the system (1).

From a practical point of view, it may be useful to consider the preceding concepts in a discrete-time setting. Because the system is monitored at a sampling period τ, the following discrete-time representation of Eq. (4) can be considered:

$$Y_k = H_k Y_{k-1}, \qquad H_k \in R^{q,q} \tag{10}$$

with

$$Y_k = [\boldsymbol{y}_k | \boldsymbol{y}_{k-1} | \cdots | \boldsymbol{y}_{k-q+1}] \tag{11}$$

$$Y_{k-1} = [\boldsymbol{y}_{k-1} | \boldsymbol{y}_{k-2} | \cdots | \boldsymbol{y}_{k-q}] \tag{12}$$

As for the continuous-time setting, we may consider the eigenvalues of H_k at each sampling instant k, so that the corresponding eigenvalues can be obtained. In cases of ill conditioning of Y_{k-1}, one may include more measurement instants than sensors and use pseudoinversion to obtain H_k as in Eq. (6).

Eigenvalue Orbits for DLTI Systems

This section illustrates how an exact formulation of the eigenorbit equations may be obtained in the case of free response of a known DLTI system with noise-free measurements. The formalism established in this section shall serve subsequently as basis for the demonstration of the basic eigenvalue orbit theorems.

Consider the following DLTI system under initial conditions:

$$\boldsymbol{x}_k = A^k \boldsymbol{x}_0, \qquad \boldsymbol{y}_k = C\boldsymbol{x}_k \tag{13}$$

with $A \in R^{n,n}$ and $C \in R^{q,n}$ and $k = 1, 2, \ldots$. To obtain matrix H_k for this system, it is necessary to construct matrices Y_k and Y_{k-1} as in Eqs. (11) and (12). In this case, these matrices can be readily written in the form

$$Y_k = CA^k\left[\boldsymbol{x}_0 \middle| A^{-1}\boldsymbol{x}_0 \middle| \cdots \middle| A^{1-q}\boldsymbol{x}_0\right] = CA^k Q_q \tag{14}$$

$$Y_{k-1} = CA^{k-1}\left[\boldsymbol{x}_0 \middle| A^{-1}\boldsymbol{x}_0 \middle| \cdots \middle| A^{1-q}\boldsymbol{x}_0\right] = CA^{k-1} Q_q \tag{15}$$

where $Q_q = [\boldsymbol{x}_0 | A^{-1}\boldsymbol{x}_0 | \cdots | A^{1-q}\boldsymbol{x}_0]$, and which yields

$$CA^k Q_q = H_k CA^{k-1} Q_q \tag{16}$$

This last equation may be inverted as follows:

$$H_k = CA^k Q_q \left[CA^{k-1} Q_q\right]^{-1} \tag{17}$$

Note that, whereas A has n constant eigenvalues, H_k has q time-dependent eigenvalues. From the particular structure of this last equation, information on the eigenorbits can be obtained. In fact, the preceding formulation enables the computation of the exact orbits for a known DLTI system and with a given initial state vector $\boldsymbol{x}_0$. In a more general case, when the only information about the system is obtainable through a set of sensors, the orbits can still be computed experimentally. To this end, it is sufficient to assemble the matrices Y_k and Y_{k-1} and to compute $H_k = Y_k Y_{k-1}^{-1}$. Once H_k is available, its eigenvalues may be obtained and the corresponding orbits updated. Having obtained the eigenvalues, the data composing matrices Y_k and Y_{k-1} are updated so that a new H_k may be computed. The process may be repeated indefinitely. It is important to note that, although the basic continuous-time definition of $H(t)$ requires derivatives of the sensor readings $\dot{y}$, the discrete formulation introduced requires only sampled measurements in a practical (experimental) orbit determination. The discrete-time approach, therefore, eliminates the necessity of sensor output differentiation.

Properties of Eigenvalue Orbits for DLTI Systems

In this section the formulation is exploited to state and prove theorems characterizing the eigenvalue orbits for the following prototype system with noise-free measurements:

$$\mathcal{S}_f : \boldsymbol{x}_k = A^k \boldsymbol{x}_0, \qquad \boldsymbol{y}_k = C\boldsymbol{x}_k \tag{18}$$

with $A \in R^{n,n}$ and $C \in R^{q,n}$. $\mathcal{S}_f$ is the generating system. The associated system is

$$\mathcal{S}_l : Y_k = H_k Y_{k-1} \tag{19}$$

with $H_k \in R^{q,q}$.

Definition 1: An eigenorbit, or λ orbit, is the trajectory described by an eigenvalue of H_k on the complex plane along time.

Matrix H_k possesses q eigenorbits. The ith orbit, corresponding to eigenvalue λ_i, is $O^{(i)}$, whereas $\lambda_k^{(i)}$ is a point on that orbit at the kth instant.

Any conclusions regarding $\mathcal{S}_l$ that may be drawn from the orbit examination, such as stability, stochasticity, conservativeness, etc., apply also to $\mathcal{S}_f$ because the associated system $\mathcal{S}_l$ is, by definition, an output equivalent realization of the generating system $\mathcal{S}_f$.

Definition 2: The ith λ orbit is said to be degenerate if

$$\lambda_k^{(i)} = \lambda_0^{(i)}, \qquad \forall k > 0 \tag{20}$$

In all other cases, the orbit is nondegenerate.

Definition 3: The ith λ orbit is periodic (closed) if there exists a constant and finite M such that $\lambda_k^{(i)} = \lambda_{k+M}^{(i)}$ for every positive k.

In this case, M is the orbit period. It is clear that if all of the orbits of the system $\mathcal{S}_l$ are periodic, then also the matrix H_k is periodic and so is system $\mathcal{S}_f$.

In the sequel, the basic eigenorbit theorems shall be formulated and proved. The first theorem addresses the conditions of existence of degenerate eigenorbits. Further theorems address issues such as orbit periodicity, shape, and stability.

Theorem 1: All of the eigenorbits of the DLTI system $S_f : \boldsymbol{x}_k = A^k \boldsymbol{x}_0$ are degenerate if $q = n$ and if the system is observable.

Proof: Under the assumption that $q = n$ and with noise-free measurements, it is immediately clear that $Q_q \in R^{n,n}$ and $C \in R^{n,n}$. Therefore, the equation defining H_k

$$H_k = CA^k Q_q \left[CA^{k-1} Q_q\right]^{-1} \tag{21}$$

becomes

$$H_k = CA^k Q_n Q_n^{-1} A^{1-k} C^{-1} = CAC^{-1} = H \tag{22}$$

Consequently, by virtue of the properties of the similarity transformation, both H and A have the same eigenspectrum, which is, of course, time invariant. Therefore, when $q = n$, i.e., when the number of sensors equals the number of active states, the eigenorbit algorithm identifies simply a state matrix H whose eigenvalues constitute the degenerate orbits. Observability of the system guarantees the existence of the inverse of C. □

It is important to note that if $q > n$, that is, when there are more sensors than active states, H_k becomes rank deficient with $m = q - n$ null eigenvalues. In this case, H_k must be calculated according to

$$H_k = Y_k Y_{k-1}^{+} \tag{23}$$

where the pseudoinversion may be performed according to the singular value decomposition. In case of noise-corrupted measurements, the rank of H will, in general, be full. Therefore, the algorithm will identify, together with the physical states of the system, additional modes corresponding to noise. The effect of noise in the measurement loop is reflected by an evident stochastic character of the orbit (fuzziness), as will be shown in Example 2.

The following theorem addresses conservative systems and their orbits.

Theorem 2: The eigenvalue orbits of a conservative DLTI $S_f : \boldsymbol{x}_k = A^k \boldsymbol{x}_0$ system are all periodic.

Proof: Because the spectrum of a periodic matrix is periodic as well, the theorem will be proved if H_k can be shown to be periodic. According to the definition of H_k, i.e.,

$$H_k = Y_k Y_{k-1}^{-1} \tag{24}$$

H_k is periodic iff Y_k and Y_{k-1} are periodic. Because Y_k and Y_{k-1} have identical structure, the proof shall regard only periodicity of Y_k. This periodicity implies that a finite N exists for which the following holds:

$$Y_k = Y_{k+N} \tag{25}$$

This is equivalent to

$$CA^k Q_q = CA^{k+N} Q_q \tag{26}$$

This is true iff

$$A^k = A^{k+N} \tag{27}$$

This may be seen as

$$A^{k-1}A = A^{k-1}AA^N \tag{28}$$

which finally leads to

$$A = A^{1+N} = A^M \tag{29}$$

A matrix satisfying the preceding relation is called idempotent. This property is reserved for matrices with eigenvalues equal to either 0 or 1. The former condition is precisely the case of A, whose eigenvalues lie on the unit circle, by virtue of the system's conservative nature. The idempotency of A is, of course, equivalent to

$$\boldsymbol{x}_M = \boldsymbol{x}_0 \tag{30}$$

which can only be satisfied if the system is conservative. The theorem, therefore, has been proved. □

If a general nonlinear system is conservative, then the associated linear time-varying (LTV) system described by H_k is also periodic. Therefore, the validity of Theorem 2 may be extended to any nonlinear generating system.

The following theorem proves the existence of nondegenerate eigenorbits when the number of states exceeds the number of sensors. It is, therefore, fundamental for the theory presented herein in that it shows that the eigenorbits are indeed a unique proper physical property that one may attribute to any generating system.

Theorem 3: The DLTI system $\mathcal{S}_f$ has nondegenerate λ orbits if $n > q$ and if A is not idempotent.

Proof: The theorem shall be proved if H_k can be shown to vary with time k. Let us perform the spectral decomposition of the state matrix

$$A = TJT^{-1} \tag{31}$$

where J is the Jordan form of A and T is the modal matrix. Excluding repeated eigenvalues or other pathological eigenstructures, J will be diagonal. In the most general case, the Jordan form of a matrix is block diagonal, and the subsequent argument is similar. In these conditions, H_k becomes

$$H_k = CTJ^kT^{-1}Q_q\left(CTJ^{k-1}T^{-1}Q_q\right)^{-1} \tag{32}$$

which yields

$$H_k = RJ^kW(RJ^{k-1}W)^{-1} = f_1(J^k)f_2^{-1}(J^{k-1}) \tag{33}$$

where $R = CT \in R^{q,n}$ and $W = T^{-1}Q_q \in R^{n,q}$ are constant matrices. Equation (33) indicates that, if H_k depends on k, then this dependence is a consequence of the time dependence of the powers of the Jordan forms. The nature of this dependence may be investigated in the following manner. The entries of J are, in general, complex numbers of the type $\mu_i = \rho_i e^{j\theta_i}$ with $i = 1, 2, 3, \ldots, n$ and $j = \sqrt{-1}$. Therefore, calculation of the powers of J is equivalent to the calculation of the powers of each of the eigenvalues of A. This may be done with the aid of Euler's theorem as follows (the index i is dropped for brevity of notation):

$$\mu^k = \rho^k e^{jk\theta} = \rho^k(\cos k\theta + j\sin k\theta) \tag{34}$$

Equation (34) indicates a very important fact, namely, that the elevation of J to powers of k is equivalent to a progressive rotation in the complex plane of the vectors defined by the eigenvalues of A. Therefore, depending on the modulus of each of the eigenvalues ρ_i, this rotation, defined clearly by the term $k\theta$, is accompanied by either a divergence or convergence of μ_i^k provided μ_i is neither 0 nor 1 because A is not idempotent. Consequently, J^k and J^{k-1} both vary with k and $J^k \neq J^{k-1}$. It now remains to prove that $f_1(J^k)f_2^{-1}(J^{k-1})$ is not constant. Because $f_1 = f_2$, the only condition under which $H_k = f_1(J^k)f_2^{-1}(J^{k-1})$ can be constant is $J^k = J^{k-1}$. This condition, again, requires A to be idempotent, a fact which is excluded by the hypothesis of the theorem. □

The following theorem addresses the important issue of orbit stability and how it relates to the stability of the generating system $\mathcal{S}_f$.

Theorem 4: The eigenorbits associated with a dissipative DLTI $\mathcal{S}_f$ system with $n > q$ tend to degenerate orbits.

Proof: Let A_n be the matrix of system $\mathcal{S}_f$ at the initial instant $k = 0$. Suppose that A_n is already in modal form. If, for $k = 0$, all modes have either nonzero initial displacement or velocity, then, given that $\mathcal{S}_f$ is dissipative, it is always possible to find such N_1 that for $k > N_1$ some mode has decayed and, therefore, no longer contributes to the λ orbits of H_k. Another consequence of this is that A_n may now be deflated down to A_{n-2}. Similarly, increasing k, it is possible to find such an N_2 that for $k > N_2$ some other mode vanishes and no longer contributes to the sensor readings. Therefore, now one has $A \in R^{n-4,n-4}$. One may continue this reasoning until $A \in R^{q,q}$, in which case the conditions of Theorem 1 are met and all orbits become degenerate. The norm of H, now constant, is $\|CA_qC^{-1}\| = \bar{\sigma}(CA_qC^{-1})$, where $\bar{\sigma}()$ is the largest singular value of ( ). This last fact proves the theorem. □

It is interesting also to examine the situation when there are more sensors than states, i.e., $q > n$. Continuing the reasoning used earlier, it is clear that this condition is reached once enough modes have decayed to make Y_k rank deficient. This rank deficiency of Y_k propagates onto H_k, which exhibits, at this stage, $q - n$ null eigenvalues. In the limit case of null sensor readings, i.e., $Y_k \equiv [\varnothing]$, one obtains also $H_k \equiv [\varnothing]$, which possesses one null eigenvalue with multiplicity q that corresponds to a particular degenerate orbit. In this case, one may draw the conclusion that all of the sensors are measuring the same rigid-body mode, in the sense of a drift of all of the state variables of a given dynamical system.

The preceding theorem states in practice that if all of the λ orbits of the associated system $\mathcal{S}_l$ tend to a set of degenerate orbits, then the generating system $\mathcal{S}_f$ is necessarily stable (dissipative). This is very important because in the general case it is impossible (and often unnecessary) to obtain information on the structure of $\mathcal{S}_f$. This theorem suggests also that the control of a generic and unknown system $\mathcal{S}_f$ may be accomplished if one can control the λ orbits in some appropriate manner. This particular form of control via λ orbits shall not be addressed in this paper, being currently the subject of research.

It must be remarked that, although the preceding theorems have been formulated for a simple DLTI system ($\boldsymbol{x}_k = A^k\boldsymbol{x}_0, \boldsymbol{y} = C\boldsymbol{x}$), their scope may be suggested to more general nonlinear systems with unknown structure and dimension. It is known that any nonlinear system, having (locally) a Jacobian that does not possess purely imaginary eigenvalues, may be studied (locally) in terms of stability, via the associated linearized system. The dynamic matrix of this system is, precisely, the Jacobian. This is expressed via the Hartman–Grobman theorem. Therefore, any nonlinear system satisfying this condition may be approximated, along its trajectory, by a series of linear time-invariant (LTI) systems, each of which is valid only in a small vicinity of each point along the trajectory and for a small amount of time. Now, because each of these linear systems will differ from its predecessor by a small amount, one may approximate this sequence of LTI systems by an equivalent LTV system. The fundamental property of eigenorbits is that they do not discriminate between an LTI system that has more active states than sensors ($q > n$) or an LTV system in which $q = n$. Both such systems lead to nondegenerate eigenorbits which, under certain circumstances, may even be identical. In systems of the type $\dot{\boldsymbol{x}} = A\boldsymbol{x} + \epsilon f(\boldsymbol{x})$, where $\epsilon \ll 1$ with $f(\boldsymbol{x})$ being nonlinear function, the eigenorbits will tend to those of the linear system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ as ϵ gets smaller.

Numerical Examples

In this section, four examples have been selected with the purpose of providing a more intuitive and closer look at the salient properties of eigenorbits.

Example 1

Consider a simple DLTI four-degree-of-freedom (4-DOF) system. This system is assumed to play the role of the unknown but observable reality, i.e., $\mathcal{S}_f$. The system has been obtained by sampling the corresponding continuous system at 40 Hz and has the following form:

$$\boldsymbol{x}_k = A_4\boldsymbol{x}_{k-1}, \qquad \boldsymbol{y}_k = C_4\boldsymbol{x}_k \tag{35}$$

with

$$A_4 = \begin{bmatrix} 0.9997 & 0.0250 & 0 & 0 \\ -0.0250 & 0.9994 & 0 & 0 \\ 0 & 0 & 0.9991 & 0.0249 \\ 0 & 0 & -0.0747 & 0.9916 \end{bmatrix}$$
$$C_4 = \begin{bmatrix} 0 & 0.1 & 0 & -0.1 \\ 1 & 1 & 0 & 0.4 \end{bmatrix} \tag{36}$$

with the following initial state vector $\boldsymbol{x}_0 = \{1, 0, 0.01, 0\}^T$. The eigenvalues of A_4 are $\{\lambda_{1,2} = 0.9996 \pm j0.0250, \lambda_{3,4} = 0.9953 \pm j0.0430\}$. Three time instants are considered at each step k, i.e., k, $k-1$, and $k-2$, and $H_k \in R^{2,2}$ because there are two sensors. The orbits of the two eigenvalues are complex conjugate, just like the eigenvalues of A. Figure 1 shows the first of these orbits. The orbit starts at $t = 0$ as indicated in the figure by an asterisk and spirals toward the focal point indicated as +. This point coincides with

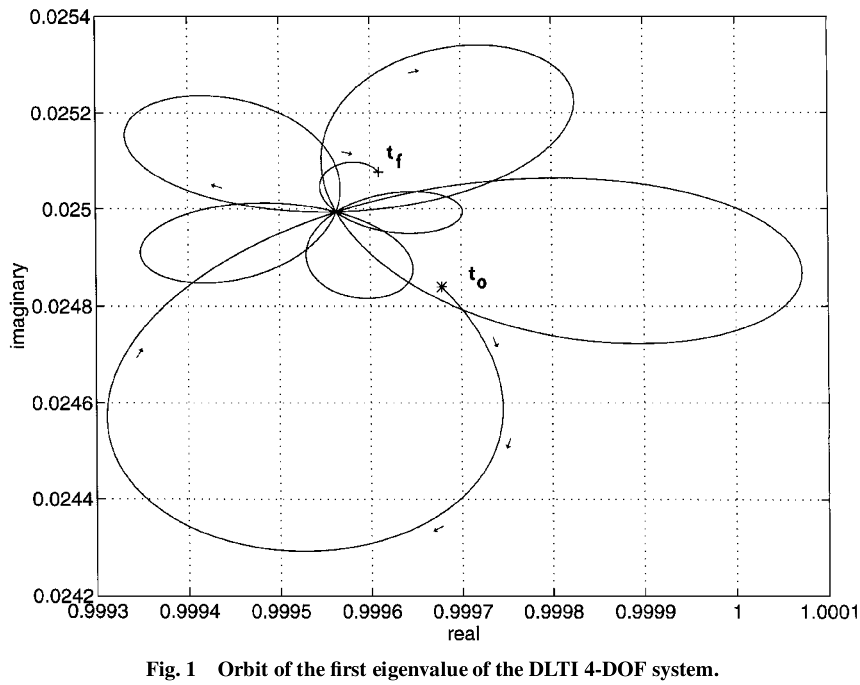

Fig. 1 Orbit of the first eigenvalue of the DLTI 4-DOF system.

the location of the slow mode, i.e., $0.9996 \pm j0.0250$. As the fast mode decays due to internal dissipation, it may be observed that the generating DLTI system $\mathcal{S}_f$ converges toward the following 2×2 one:

$$A_2 = \begin{bmatrix} 0.9997 & 0.0250 \\ -0.0250 & 0.9994 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 0 & 0.1 \\ 1 & 1 \end{bmatrix} \tag{37}$$

In this simple case, because C_2 is square and has full rank, one may easily verify that both A_2 and H are similar matrices:

$$H = C_2 A_2 C_2^{-1}, \qquad H \in R^{2,2} \tag{38}$$

with

$$H = \begin{bmatrix} -1.0244 & -0.0025 \\ 0.4951 & 0.9749 \end{bmatrix} \tag{39}$$

and having the same spectrum as A_2. This is clearly understood in terms of Theorem 4, which states that the orbits of dissipative systems tend to degenerate orbits. System (35) is dissipative, and so its orbit tends to points, as may be observed in Fig. 1.

Example 2

Although the theory presented in the preceding sections assumes no sensor noise, some interesting features can be outlined from a numerical example in the presence of sensor noise. In case of noise-corrupted measurements, the system equations assume the following form:

$$\boldsymbol{x}_k = A_4 \boldsymbol{x}_{k-1}, \qquad \boldsymbol{y}_k = C_4 \boldsymbol{x}_k + \boldsymbol{v}_k \tag{40}$$

where v is a stochastic process. The direct implication of this source term is the perturbation of the Y_k and Y_{k-1} matrices. There are two important consequences of the presence of measurement noise, namely, 1) perturbation of the eigenspectrum of H and 2) compensation of any rank deficiencies in H.

The noise terms in fact constitute a random perturbation of H, which can, under certain circumstances, be rank deficient. However,

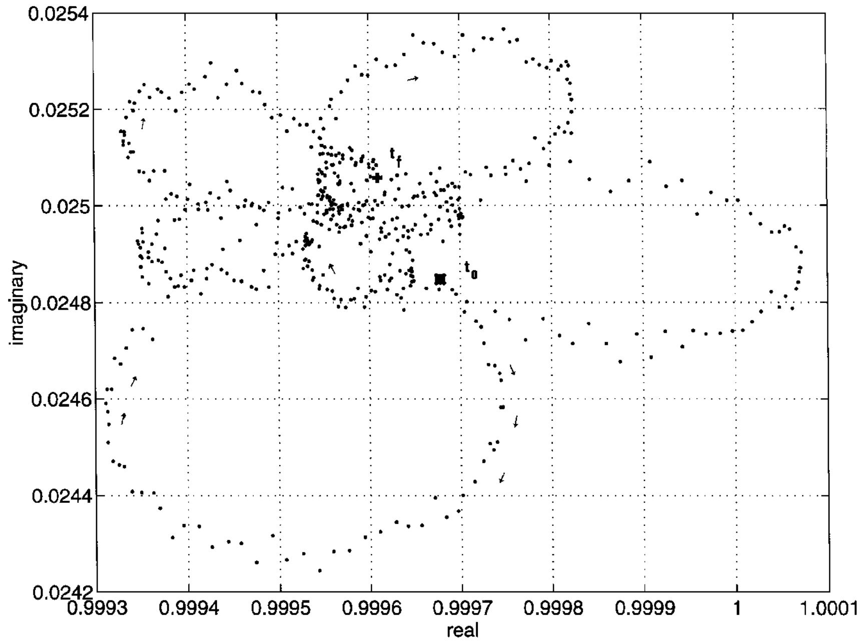

Fig. 2 Stochastic orbit of the 4-DOF DLTI system with noisy measurements.

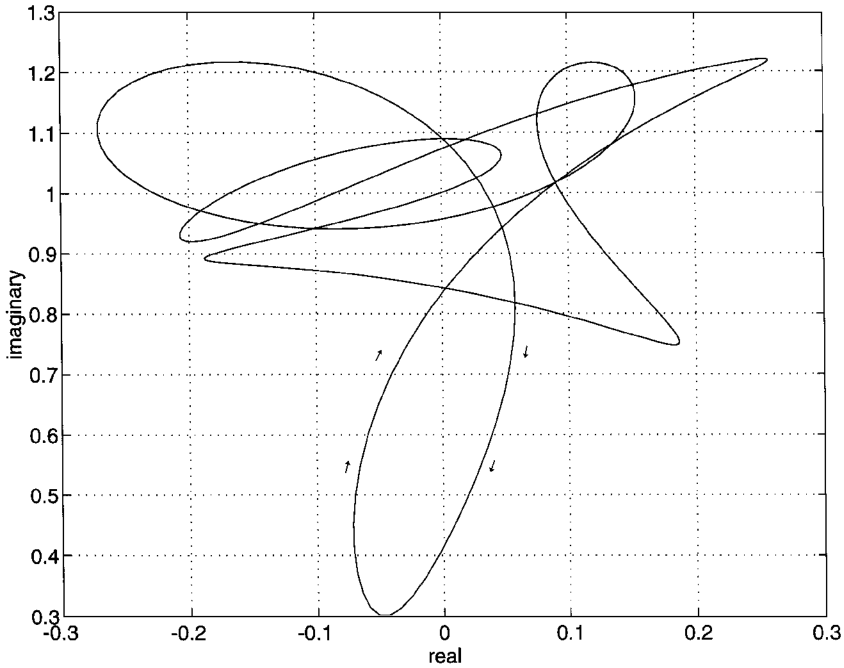

Fig. 3 Eigenorbit of a 10-state LTI system.

adding to H a random matrix makes up for this deficiency because any random matrix is almost certainly nonsingular. This is true especially for large matrices, which exhibit extremely low probabilities of singularity. The important issue is to realize that the measurement noise is not falsifying conclusions about dynamics of the system because the noise itself is part of the process. The effects of measurement noise are clearly visible in Fig. 2, where the pronounced fuzziness of the orbits, although still retaining the original deterministic form, suggests that perhaps a very fast state (noise) is present within the system. Alternatively, one could also conclude that the orbit could be generated by a system with stochastic parameters and without noise. In any case, this distinction is not very important as long as the orbit remains stable.

Example 3

The central issue of the theory presented in this paper is that λ orbits reflect the state of a system or a process. The complexity of the orbit shape depends essentially on the complexity of the generating system and on the number of sensors used for monitoring. Figures 3 and 4 show examples of complex orbits of 10- and 20-state LTI systems, whereas Figs. 5 and 6 show the first two orbits of a simpler 6-state linear system. In all cases the eigenorbits have been obtained with four sensors. The clear characteristic that may be quickly inferred by glancing at the plots is that they all show closed curves. This denotes a cyclic and stable behavior of the respective generating systems. According to Theorem 2 this type of closed orbit can be attributed to conservative systems. The lack of symmetry and the intricate shapes, especially in the orbits in Figs. 3 and 4, reflect that the corresponding generating systems are composed of numerous states. In fact, the difference in the number of states and the number of sensors used to obtain the orbits is large in both these cases, namely, 20 and 10 vs 4. In Figs. 5 and 6 this difference is much smaller, i.e., 6 states vs 4 sensors, which is reflected in a less intricate shape of the orbits.

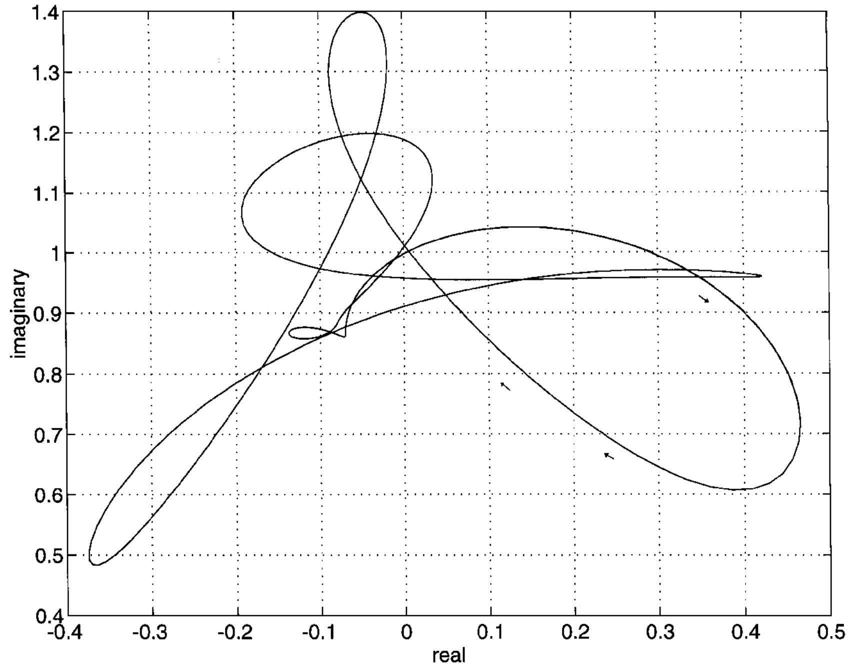

Fig. 4 Eigenorbit of a 20-state LTI system.

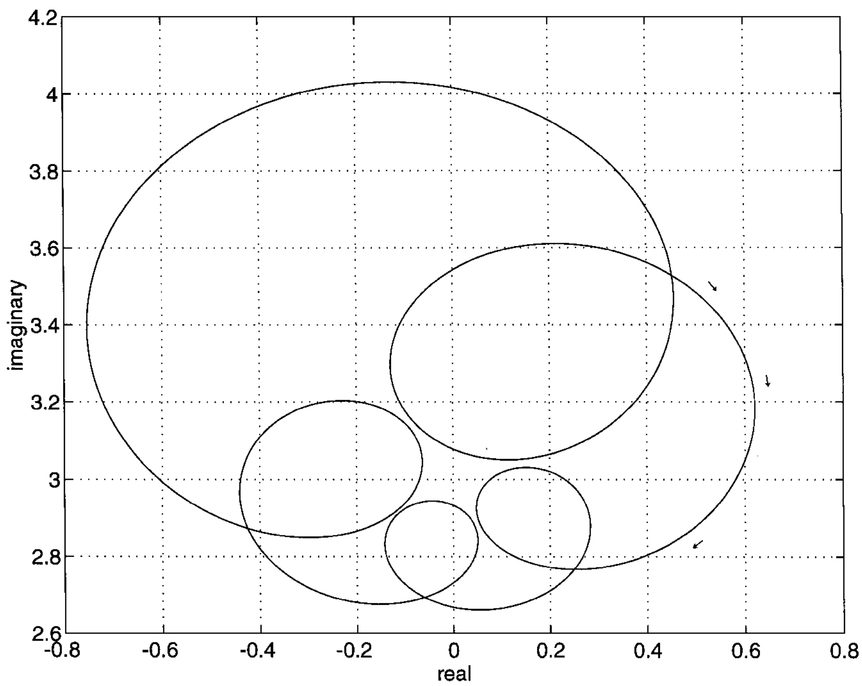

Fig. 5 First eigenorbit of a 6-state LTI system.

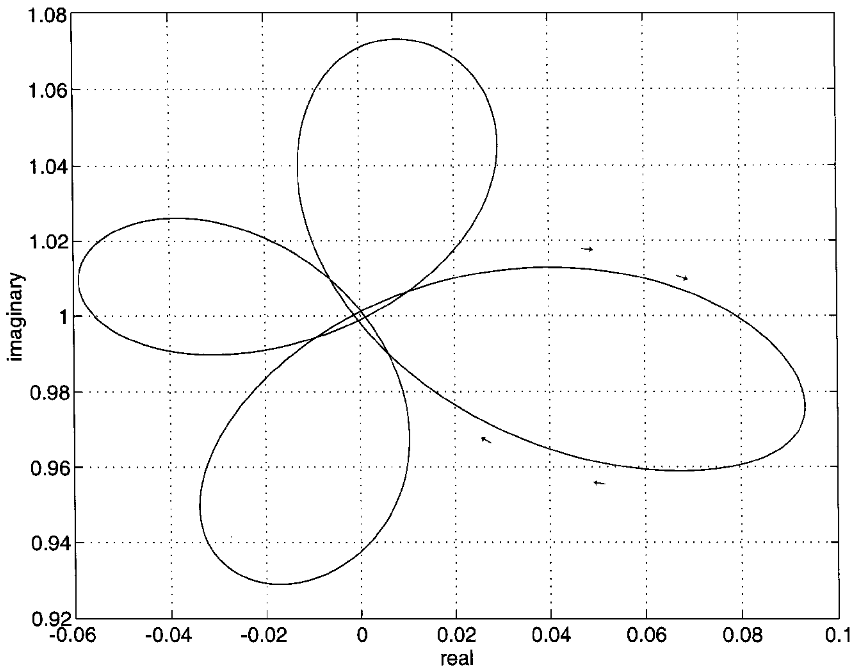

Fig. 6 Second eigenorbit of a 6-state LTI system.

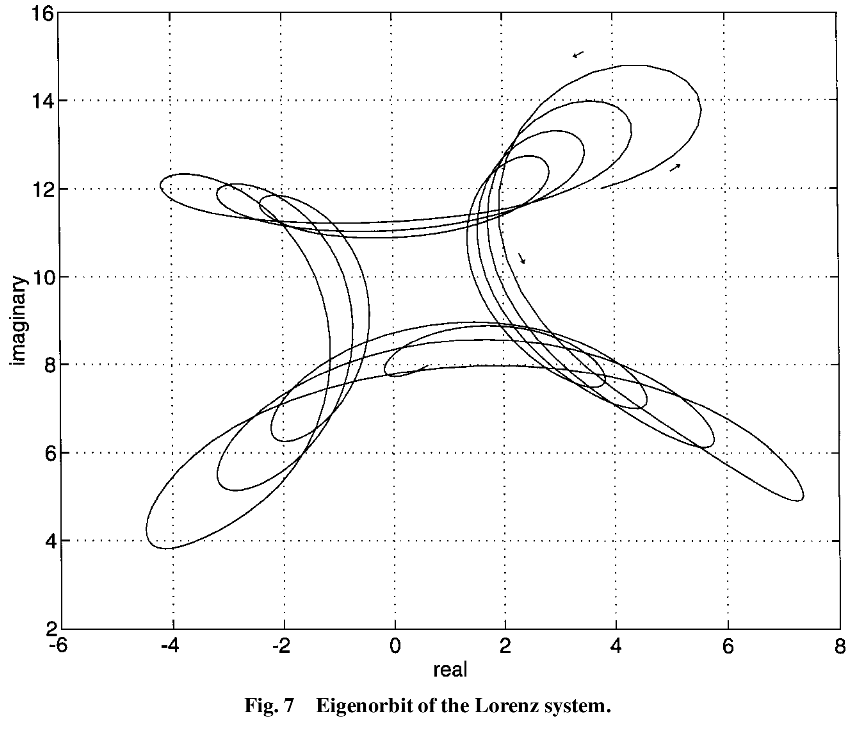

Fig. 7 Eigenorbit of the Lorenz system.

Example 4

As advanced in the preceding sections, eigenorbits may be used to provide information on nonlinear systems. An example of eigenorbits of the Lorenz system is shown in Fig. 7. The Lorenz system is described by the following coupled quadratic ordinary differential equations:

$$\dot{x} = \sigma(y - x), \qquad \dot{y} = \rho x - y - xz, \qquad \dot{z} = -\beta z + xy \quad (41)$$

with $\sigma, \rho, \beta > 0$ and $x, y, z \in R$. These parameters represent, respectively, the Prandtl number, the Reynolds number, and an aspect ratio. The original values of $\beta = \frac{8}{3}$ and $\sigma = 10$ used by Lorenz have been adopted, together with $\rho = 28$. The complete phase portrait has been obtained with $\Delta t = 0.005$ s and 2700 iterates. The apparently simple phase portrait reveals its complexity in the corresponding λ orbit. It is interesting to relate local instabilities in the phase portrait to portions of the λ orbit that reside in the right-hand plane. The phase portrait shows how the spiral's radius increases, thereby revealing an evident global instability. There are, however, portions of the spiral that actually exhibit local convergence, which is reflected in the corresponding λ orbit. The orbit does, of course, confirm that the system is globally in a state of divergence because its radius also increases. The fact that more than half of the λ orbit resides in the right-hand plane leads to the conclusion that there are more local instabilities than stabilities in the phase portrait. In other words, there are more segments of the phase portrait that appear to diverge locally than those exhibiting local convergence. Moreover, observing the orbit, one may conclude that its center of gravity (focal point) is located approximately in $(0.5 + 10j)$, which can be viewed as a sort of equivalent or average root. This average root, just like the entire orbit, is of course unstable and reflects the average global behavior of the orbit. Finally, it is important to remark that the underlying complexity of the spiral's dynamics is not at all evident in its phase portrait. The λ orbit, on the other hand, exposes the intricate nature of the Lorenz system, which would otherwise be visible if one resorted to higher derivatives of the phase portrait.

Remarks

These examples, together with the developed theory, suggest the conclusion that nondegenerate λ orbits may be used to detect instabilities, the presence of nonlinearities, the presence of more states than sensors (a form of spillover), the presence of control, the presence of external excitation, the time-dependence of the system parameters, or any combination thereof.

Bearing in mind these considerations, a possible approach to the problem of process control could be one where a controller is used to steer the dynamics of the orbits instead of the states (phase trajectories), thereby accomplishing some specific control objectives.[9,11] One particularly obvious application of eigenorbits in this context is, for example, to provide a time-switching law for classical controllers. It is clear that when an orbit crosses the imaginary axis, somewhere in the system some sort of instability is appearing. This instability lasts for all of the time that the orbit resides in the complex right-hand plane. Therefore, a controller may be activated when the orbit approaches the imaginary axis from the left-hand plane and deactivated when the orbit migrates back into the left-hand plane. This approach to control has a predictive potential in that orbit monitoring provides a complete picture of the system's past and of its short-term tendencies. Preliminary numerical experiments suggest that this concept is indeed viable, and research in this direction is currently under way.

Conclusions

The concept of λ orbit has been introduced, and basic theorems illustrating the orbit properties have been proved for a DLTI system under free response. Numerical examples have illustrated how the orbit shapes can be used for system and process characterization and monitoring. It has also been shown how the eigenvalue orbits of the associated system provide insight into the dynamic characteristics of the generating system. Being a physical attribute of a dynamic system, λ orbits do not depend, in a reasonable range, on the sensor output sampling frequency. A numerical procedure has also been presented for the practical computation of λ orbits via sampling of the available sensor readings. Finally, the orbits have been shown to have potential control applications in that they reflect the state of dynamical processes or systems.

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